5 RATIO & PROPORTION

If a and b ($b \neq 0$) are two quantities of the same kind, then Ratio is the relation which one quantity bears to another of the same kind in magnitude.

Now in two quantities a and b the fraction $\frac{a}{b}$ is called the ratio of a to b. It is usually expressed as a : b, a and b are said to be the terms of the ratio. The former (numerator) ‘a’ is called the ***Antecedent*** of the ratio and latter (denominator) ‘b’ is called ***consequent***.

Properties in Ratio

(i) As ratio is a relation between two quantities so ratio is independent of the concrete units employed in the quantities compared.

(ii) Ratio exists only between two quantities, both the quantities must be in the same units.

Composition of Ratio

I. Compounded Ratio

When two or more ratios are multiplied term wise, the ratio thus, obtained is called their compounded ratio.

Example: a : b, c : d, e : f

Compounded ratio = $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ace}{bdf}$

II. Duplicate Ratio

It is the compounded ratio of two equal ratios.

Thus the duplicate ratio of a : b = $\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$.

Hence duplicate ratio of a : b = $a^2 : b^2$

> ***Do you know?***
>
> **(i)** To compare two ratios in fraction, make the denominator or numerator equal of both the ratios.
>
> **(ii)** If the terms of the ratio are multiplied or divided by the same number, then the value of the ratio will not undergo any change.

III. **Triplicate Ratio**

It is the compounded ratio of three equal ratios.

Thus the triplicate ratio of a : b = $\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$

Hence triplicate ratio of a : b = $a^3 : b^3$

IV. **Sub-duplicate Ratio**

For any ratio a : b, its sub-duplicate ratio is defined as $\sqrt{a} : \sqrt{b}$

V. **Sub-triplicate Ratio**

For any ratio a : b its sub-triplicate ratio is defined as $\sqrt[3]{a} : \sqrt[3]{b}$

VI. **Reciprocal Ratio**

For any ratio a : b, b ≠ 0 the reciprocal ratio will be $\frac{1}{a} : \frac{1}{b}$

Example

Find the number, which has the same ratio to $\frac{5}{9}$ as $\frac{2}{7}$ has to $\frac{3}{28}$.

Sol. Let the required number = x

Then, according to Question

$$x : \frac{5}{9} = \frac{2}{7} : \frac{3}{28}$$

$$\Rightarrow x \times \frac{9}{5} = \frac{2}{7} \times \frac{28}{3}$$

$$\Rightarrow x \times \frac{9}{5} = \frac{8}{3}$$

$$\Rightarrow x = \frac{8}{3} \times \frac{5}{9} = \frac{40}{27}$$

Hence $x = \frac{40}{27}$.

Example

A ratio is equal to 3 : 5. If the antecedent is 7, find the consequent.

Sol. Given ratio = 3 : 5; Antecedent = 7

Let consequent = x

Now $\frac{3}{5} = \frac{7}{x}$

$\Rightarrow 3x = 35 \quad \Rightarrow x = \frac{35}{3}$

Proportion

When two ratios are equal, the four terms involved, taken in order are called proportionals, and they are said to be in proportion.

1. The ratio of a to b is equal to the ratio of c to d i.e. iff $\frac{a}{b} = \frac{c}{d}$, we write a : b = c : d

2. When a : b = c : d we write as $\frac{a}{b} = \frac{c}{d}$

Here, the terms a and d are called Extremes and, the terms b and c are called the Means. The fourth term 'd' is called Fourth Proportional to a, b, c.

Continued Proportion:

Three quantities are said to be in continued proportion, if the ratio of the first to the second is same as the ratio of the second to the third.

Thus a, b and c are in continued proportion if a : b = b : c

$$\frac{a}{b} = \frac{b}{c}$$

Mean Proportion:

If a, b, c are in continued proportion then second quantity 'b' is called the mean proportional between 'a' and 'c'.

$\frac{a}{b} = \frac{b}{c}$ then mean proportion $b^2 = ac$

or $b = \sqrt{ac}$

Third Proportional:

When a, b, c are in continued proportion then the third quantity 'c' is called the third proportional to 'a' & 'b'

$$\frac{a}{b} = \frac{b}{c}$$

Third proportional $c = \frac{b^2}{a}$.

Example

Find the mean proportional to

(a) 3, 27 ***(b)*** $\mathbf{\frac{5}{36}, \frac{3.2}{25}}$

Sol. (a) Let the mean proportion of 3, 27 be x

$\therefore$ 3 : x = x : 27 $\Rightarrow \frac{3}{x} = \frac{x}{27}$

$x^2 = 3 \times 27$, $x = \sqrt{3 \times 27}$, $x = \sqrt{81}$ $\Rightarrow x = 9$.

(b) Let the mean proportion of $\frac{5}{36}$ and $\frac{3.2}{25}$ be x

$\therefore$ $\frac{5}{36} : x = x : \frac{3.2}{25}$ $\Rightarrow \frac{5}{36} \times \frac{1}{x} = x \times \frac{25}{3.2}$

$x^2 = \frac{5}{36} \times \frac{3.2}{25} = \frac{16}{900}$

$x = \sqrt{\frac{16}{900}} = \frac{4}{30} = \frac{2}{15}$ $\Rightarrow x = \frac{2}{15}$.

Example

What should be added to each of the numbers 19, 26, 37, 50 so that the resulting number may be in proportion?

Sol. Let the required number be x.

According to question,

19 + x, 26 + x, 37 + x and 50 + x are in proportion

$19 + x : 26 + x = 37 + x : 50 + x \quad \Rightarrow \frac{19+x}{26+x} = \frac{37+x}{50+x}$

$(19 + x)(50 + x) = (37 + x)(26 + x) \quad \Rightarrow x^2 + 69x + 950 = x^2 + 63x + 962$

$x^2 - x^2 + 69x - 63x = 962 - 950$

Hence the resulting number = 2.

Example

If q is the mean proportional between p and r, prove that

$$p^2 - q^2 + r^2 = q^4\left(\frac{1}{p^2} - \frac{1}{q^2} + \frac{1}{r^2}\right).$$

Sol. Given q is the mean proportional between p and r

$\therefore \quad q^2 = pr$

L.H.S. $= p^2 - q^2 + r^2$

R.H.S. $= q^4\left(\frac{1}{p^2} - \frac{1}{q^2} + \frac{1}{r^2}\right)$ putting $q^2 = pr$

$= (pr)^2\left(\frac{1}{p^2} - \frac{1}{q^2} + \frac{1}{r^2}\right) = p^2r^2\left(\frac{1}{p^2} - \frac{1}{q^2} + \frac{1}{r^2}\right)$

$= \left(\frac{p^2r^2}{p^2} - \frac{p^2r^2}{q^2} + \frac{p^2r^2}{r^2}\right) = \left(\frac{p^2r^2}{p^2} - \frac{p^2r^2}{pr} + \frac{p^2r^2}{r^2}\right)$

$= (r^2 - pr + p^2) \quad \Rightarrow r^2 - q^2 + p^2 \quad [pr = q^2]$

$= p^2 - q^2 + r^2 \Rightarrow$ L.H.S.

Hence LHS = RHS.

Properties of Proportion

1. Invertendo

If four quantities be in proportion they stay in proportion even when they are taken inversely.

If a : b : : c : d then b : a : : d : c

Since $\frac{a}{b} = \frac{c}{d}$ dividing unity be each of these equal side we have

$1 : \frac{a}{b} = 1 : \frac{c}{d}$ or $\frac{b}{a} = \frac{d}{c}$ i.e. b : a : : d : c

This result is called ***Invertendo***.

2. Alternendo

If four quantities be in proportion, they remain in proportion when they are taken alternately. If a : b : : c : d then a : c : : b : d.

Since $\frac{a}{b} = \frac{c}{d}$ multiplying both sides by $\frac{b}{c}$ we get

$$\frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c} \Rightarrow \frac{a}{c} = \frac{b}{d}$$

$\therefore$ a: c:: b : d. The result is called ***Alternendo***.

3. Componendo

When four quantities are in proportion, then the first together with the second is to second, as the third together with the fourth to the fourth are also in the same proportion.

If a : b : : c : d

Then a + b : b : : c + d : d

Now given $\frac{a}{b} = \frac{c}{d}$ adding 1 to each side.

We have $\frac{a}{b} + 1 = \frac{c}{d} + 1$

$$\frac{a+b}{b} = \frac{c+d}{d}$$

a + b : b = c + d : d. This is called ***Componendo***.

4. Dividendo

When four quantities are in proportion, the excess of the first over the second is to the second is same as the excess of the third over the fourth is to the fourth.

If a : b : : c : d

then $\frac{a-b}{b} = \frac{c-d}{d}$

Now, given $\frac{a}{b} = \frac{c}{d}$

We have $\frac{a}{b} - 1 = \frac{c}{d} - 1 \Rightarrow \frac{a-b}{b} = \frac{c-d}{d}$

a – b : b = c – d : d. This operation is called ***Dividendo***.

5. Componendo and Dividendo

When four quantities are in proportion, the ratio of sum of the first and the second to their difference is same as the ratio of the sum of third and fourth to their difference.

If a : b : : c : d then a + b : a – b : : c + d : c – d

Since $\frac{a}{b} = \frac{c}{d}$ $\therefore$ $\frac{a+b}{b} = \frac{c+d}{d}$ by componendo …..(i)

and $\frac{a-b}{b} = \frac{c-d}{d}$ ……. By Dividendo …..(ii)

Dividing (i) by (ii) we get $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

This operation is called ***componendo and Dividendo***.

Example

Show that, a, b, c, d are in proportion if

(a) ***6a + 7b : 6c + 7d : : 6a – 7b : 6c – 7d***

(b) ***$ma^2 + nb^2 : mc^2 + nd^2 : : ma^2 – nb^2 : mc^2 – nd^2$***

Sol. (A) Given 6a + 7b : 6c + 7d : : 6a – 7b : 6c – 7d

To prove a : b = c : d

Now $\frac{6a+7b}{6c+7d} = \frac{6a-7b}{6c-7d}$

Or $\frac{6a+7b}{6a-7b} = \frac{6c+7d}{6c-7d}$ By Alternendo

$\frac{6a+7b+6a-7b}{6a+7b-6a+7b} = \frac{6c+7d+6c-7d}{6c+7d-6c+7d}$ by Componendo and Dividendo

$\frac{12a}{14b} = \frac{12c}{14d}$ or $\frac{a}{b} = \frac{c}{d}$ Proved

Hence a : b = c : d.

(B) $ma^2 + nb^2 : mc^2 + nd^2 : : ma^2 – nb^2 : mc^2 – nd^2$

To prove a : b = c : d

Now $\frac{ma^2+nb^2}{mc^2+nd^2} = \frac{ma^2-nb^2}{mc^2-nd^2}$

Or $\frac{ma^2+nb^2}{ma^2-nb^2} = \frac{mc^2+nd^2}{mc^2-nd^2}$ By Alternendo

$\Rightarrow \frac{ma^2+nb^2+ma^2-nb^2}{ma^2+nb^2-ma^2+nb^2}$

$= \frac{mc^2+nd^2+mc^2-nd^2}{mc^2+nd^2-mc^2+nd^2}$ By componendo and dividendo

$\Rightarrow \frac{2ma^2}{2nb^2} = \frac{2mc^2}{2nd^2} \Rightarrow \frac{a^2}{b^2} = \frac{c^2}{d^2} \Rightarrow \frac{a}{b} = \frac{c}{d}$.

Hence a : b = c : d proved.

Example

Using properties of proportion solve each for x.

$$\frac{\mathbf{x^3+3x}}{\mathbf{3x^2+1}} = \frac{\mathbf{341}}{\mathbf{91}}$$

Sol. Given $\frac{x^3+3x}{3x^2+1} = \frac{341}{91}$

By Componendo and Dividendo

$\frac{x^3+3x+3x^2+1}{x^3+3x-3x^2-1} = \frac{341+91}{341-91} \Rightarrow \frac{(x+1)^3}{(x-1)^3} = \frac{432}{250} = \frac{216}{125}$

$\frac{(x+1)^3}{(x-1)^3} = \frac{(6)^3}{(5)^3} \Rightarrow \frac{x+1}{x-1} = \frac{6}{5}$

Again by Componendo and Dividendo

$\frac{x+1+x-1}{x+1-x+1} = \frac{6+5}{6-5} \Rightarrow \frac{2x}{2} = \frac{11}{1} \Rightarrow x = 11.$

Toolkit

Partnership & Share

If there is profit in the business run by two partners A and B then,

$$\frac{\text{Amount of A's investment} \times \text{No. of months invested by A}}{\text{Amount of B's investment} \times \text{No. of months invested by B}} = \frac{\text{Profit of A}}{\text{Profit of B}}$$

Example

Saman begins business with a capital of Rs. 50,000 and after 3 months takes Manu into partnership with a capital of Rs. 75000. Three months later Amandeep joins the firm with a capital of Rs. 1, 25,000. At the end of the year the firm makes a profit of Rs. 99,495. How much of this sum should Amandeep receive?

Sol. Money invested by Saman for 12 months = Rs. 50,000

Money invested by Manu for 9 months = Rs. 75000

Money invested by Amandeep for 6 months = Rs. 1,25,000

Share of Saman: Manu: Amandeep

= 50,000 $\times$ 12 : 75,000 $\times$ 9 ; 1,25,000 $\times$ 6 = 6,00,000 : 6,75,000 : 7,50,000

= 600 : 675 : 750 = 8 : 9 : 10

Total profit = Rs. 99,495.

Profit of Amandeep = $\left(\frac{10}{8+9+10}\right) \times 99495 = \frac{10}{27} \times 99{,}495$ = Rs. 36,850.

Corollary: If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ then

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{a+c+e}{b+d+f} = \frac{a+c-e}{b+d-f}$$

Direct Proportion

If A is directly proportional to B, then as A increases B also increases proportionally or in other words the proportional change occurs in the same direction. In general when A is directly proportional to B^n, then $\left(\frac{A}{B^n}\right)$ = Constant. As in Time & Distance problems, time taken to travel a distance is directly proportional to the distance travelled when the speed is constant. This means the distance-travelled doubles if the time taken doubles provided speed remains constant. Examples of Direct proportion are Price & Expenditure (quantity remaining same), Amount of work done & no. of men required (rate of work remaining same).

Example

If 6 men can lay 8 bricks in one lay, then how many men are required to lay 60 bricks in the same time?

Sol. 6 : x : : 8 : 60

$\Rightarrow$ x = 45 men.

OR

By Unitary method

6 $\rightarrow$ 8

60 $\rightarrow$ $\frac{6 \times 60}{8}$ = 45

Indirect/Inverse Proportion

A is in indirect proportion to B if as A increases, B decreases proportionally i.e. the proportional change occurs in the opposite direction. In general if A is in indirect proportion to B^n, then (AB^n) = constant. Again as in Time & Distance problems if the speed doubles, the time taken to cover the same distance reduces to half the original time. Examples of Indirect proportion are Price & Quantity (expenditure remaining same), No. of men required & rate of work done(amount of work remaining same).

Example

If 6 men can build a wall in 9 days then 60 men can build a similar wall in ______ days.

Sol. 6 : 60 :: x : 9

$x = 9 \times 6/60$

x = 0.9 days.

<u>Remember</u>

If x is directly proportional to y,

Then x = ky

Or $\frac{x_1}{y_1} = \frac{x_2}{y_2}$

If x is inversely proportional to y, then $x \propto 1/y$

$x = k/y$ & $x_1y_1 = x_2y_2$

The key word in the above two concepts is ‘proportion’ i.e. the change (increase or decrease) is in the same ratio. Similarly, we have a concept called **Relation** wherein the quantities related either directly or indirectly vary in the same or opposite direction resp. but not in the same proportion.

Example

A can do a piece of work in 12 days, B is 60% more efficient than A. Find the number of days it takes B to do the same piece of work.

Sol. Ratio of the efficiencies is A : B = 100 : 160 = 5 : 8. Since efficiency in inversely proportional to the no. of days, the ratio of days taken to complete the job is 8 : 5

So, number of days taken by B = $\frac{5}{8} \times 12 = 7\frac{1}{2}$ days.

Example

The price per kg of potatoes has tripled. If Rakesh wants to keep his expenditure same, by what percentage does his consumption decrease?

Sol: The price has trebled. Hence, if the expenditure has to remain the same, the quantity bought must become $\frac{1}{3}$rd of the original amount. Since the price is inversely proportional to the quantity bought. So, the drop in percentage terms is $66\frac{2}{3}$%.

Basics of Fractions

(1) When a fraction has its numerator greater than the denominator, its value is greater than one. Let us call it ***greater fraction***. Whenever a number (say x) is multiplied by a ***greater fraction***, it gives a value greater than itself.

(2) When a fraction has its numerator less than the denominator, its value is less than one. Let us call it ***less fraction***. Whenever a number (say x) is multiplied by a ***less fraction***, it gives a value less than itself.

Rule of Fractions OR Unitary method

Consider the problem: ***If 6 men can do a piece of work in 30 days of 9 hours each, how many men will it take to do 10 times the amount of work if they work for 25 days of 8 hours?***

Sol: Three points arise:

(1) Less days, so more men required.
(2) Less working hours, so more men required.
(3) More work, more men.

By rule of fraction:

Step I: We look for our required unit. It is the number of men. So, we write down the number of men given in the question. It is 6.

Step II: The number of days gets reduced from 30 to 25, so it will need **more men** (Reasoning: Less days, more men). It simply means that 6 should be multiplied by a **greater fraction** because we need a value greater than 6. So, we have: $6 \times \frac{30}{25}$

Step III: Following in the same way, we see that the above figure should be multiplied by a '**greater fraction**', i.e., by $\frac{9}{8}$. So, we have: $6 \times \frac{30}{25} \times \frac{9}{8}$

Step IV: Following in the same way, we see that the above figure should be multiplied by a '**greater fraction**' i.e. by $\frac{10}{1}$. So, we have: $6 \times \frac{30}{25} \times \frac{9}{8} \times \frac{10}{1} = 81$ men.

Alligation & Mixtures

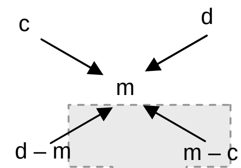
As the dictionary meaning of Alligation (mixing), we will deal with problems related to mixing of different compounds or quantities.

When two or more quantities are mixed together in different ratios to form a mixture, then ratio of the quantities of the two constituents is given by the following formulae:

$$\frac{Q_c}{Q_d} = \frac{d - m}{m - c}$$

Toolkit

Q_c	$\rightarrow$	Cheaper qty.
Q_d	$\rightarrow$	Dearer qty.
c	$\rightarrow$	C.P. of unit qty of 1st constituent.
d	$\rightarrow$	C.P. of unit qty of 2nd constituent.
m	$\rightarrow$	Mean Cost price of unit qty of mixture



$$\frac{Q_c}{Q_d} = \frac{d-m}{m-c}$$

$\frac{\mathbf{Q_c}}{\mathbf{Q_d}}$ gives us the ratio of quantities in which the two ingredients should be mixed to get the mixture.

Example

A sum of Rs 39 was divided among 45 boys and girls. Each girl get 50 paise, whereas a boy get one rupee. Find the number of boys and girls.

Sol. Average amount of money received by each = $\frac{39}{45} = \text{Rs}\frac{13}{15}$

Amount received by each girl = 50 paise = Rs $\frac{1}{2}$

Amount received by each boy = Re. 1

By alligation rule:

$$\frac{\text{Number of boys}}{\text{Number of girls}} = \frac{\frac{13}{15}-\frac{1}{2}}{1-\frac{13}{15}} = \frac{11}{4} \qquad \therefore \text{ Number of boys} = \frac{11}{11+4}\times 45 = 33$$

Number of girls = 45 – 33 = 12.

Always identify the ingredients as Cheaper & dearer to apply the *alligation rule*.

In the alligation rule, the variables c, d & m may be expressed in terms of **percentages** (e.g. A 20% mixture of salt in water), **fractions** (e.g. two-fifth of the solution contains salt) or **proportions** (e.g. A solution of milk and water is such that milk : Water : : 2 : 3) . The important point is to remember is that c & d may represent pure ingredients or mixtures.

Example

A jar contains a mixture of two liquids A and B in the ratio 4: 1. When 10 litres of the liquid B is poured into the jar, the ratio becomes 2: 3. How many litres of liquid A was contained in the jar?

Sol. Method 1:

Let the quantities of A & B in the original mixture be 4x and x litres.

According to the question $\frac{4x}{x+10} = \frac{2}{3}$.

12x = 2x + 20 $\Rightarrow$ 10x = 20 $\Rightarrow$ x = 2

The quantity of A in the original mixture = 4x = 4 $\times$ 2 = 8 litres.

Method 2 :

The average composition of B in the first mixture is 1/5.

The average composition of B in the second mixture = 1

The average composition of B in the resultant mixture = 3/5

Hence applying the rule of Alligation we have [1 – (3/5)]/[(3/5) – (1/5)] = (2/5)/(2/5) = 1

So, initial quantity of mixture in the jar = 10 litres.

And, quantity of A in the jar = $(10 \times 4)/5$ = 8 litres.

The above could also be done with liquid A. Then we need to take the averages with respect to liquid A.

Example

In two alloys, copper and zinc are related in the ratios of 4 : 1 and 1 : 3.10 kg of 1st alloy, 16 kg of 2nd alloy and some of pure copper are melted together. An alloy was obtained in which the ratio of copper to zinc was 3 : 2. Find the weight of the new alloy.

Sol. Here two alloys are mixed to from a third alloy, hence quantity of only one of the ingredients in each of the alloy will be considered.

Here, pure copper is also added, hence quantity of copper in all the three alloy will be considered.

Let the amount of pure copper = x kg.

$\therefore$ Pure copper + copper in 1st alloy + copper in 2nd alloy = copper in 3rd alloy

$$\Rightarrow \; x + \frac{4}{5}\times 10 + \frac{1}{4}\times 16 = \frac{3}{5}(10+16+x) \;\Rightarrow\; 12 + x = \frac{3}{5}(26+x)$$

$\Rightarrow$ x = 9 kg. $\therefore$ Weight of new alloy = 10 + 16 + 9 = 35 kg.

Note: In place of pure copper, if pure zinc were added, then quantity of zinc in all the three alloys have to be considered for finding the weight of the new alloy.

Toolkit

If a vessel contains "x" liters of milk and if "y" liters be withdrawn and replaced by water, then if "y" litres of the mixture be withdrawn and replaced by water, and the operation repeated 'n' times in all, then :

$$\frac{\textbf{Milk left in vessel after nth operation}}{\textbf{Initial quantity of Milk in vessel}} = \left[\frac{x-y}{x}\right]^n = \left[1-\frac{y}{x}\right]^n$$

Example

Nine litres of solution are drawn from a cask containing water. It is replaced with a similar quantity of pure milk. This operation is done twice. The ratio of water to milk in the cask now is 16 : 9. How much does the cask hold?

Sol. Let there be x litres in the cask

After n operations

$$\frac{\text{Water left in vessel after n operations}}{\text{Original quantity of water in vessel}} = (1 - 9/x)^n$$

$(1 - 9/x)^2 = 16/25 \;\therefore x = 45$ litres.